

Some characterizations of freeness of hyperplane arrangements

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Abstract

We consider some characterizations of freeness of a hyperplane arrangement, in terms of the following properties: local freeness, factorization of the characteristic polynomial, and freeness of the restricted multiarrangement. In the case of a 3-arrangement, freeness is characterized by factorization of the characteristic polynomial and coincidence of its roots with the exponents of the restricted multiarrangement. In the case of higher dimensions, it is characterized by a kind of local freeness and freeness of the restricted multiarrangement. As an application, we prove the freeness of certain arrangements conjectured by Edelman and Reiner.

1 Introduction

A hyperplane arrangement \mathcal{A} in ℓ dimensional linear space is said to be free with exponents $\exp(\mathcal{A}) = (1 = d_1, d_2, \dots, d_\ell)$ if the associated module of all logarithmic vector fields is free with basis $\delta_1, \dots, \delta_\ell$ such that $\deg \delta_i = d_i$. Among other properties we consider the following three necessary conditions for an arrangement to be free, which are due to Terao and Ziegler:

- (1) \mathcal{A} is locally free.
- (2) The characteristic polynomial $\chi(\mathcal{A}, t)$ factors completely over \mathbb{Z} , indeed, it is equal to $(t - d_1)(t - d_2) \cdots (t - d_\ell)$.
- (3) The multiarrangement obtained by restricting to a hyperplane is free with multiexponent $(d_2, \dots, d_\ell) = \exp(\mathcal{A}) \setminus \{1\}$.

It is known that each condition is not sufficient to characterize freeness. For example, since any central 2-(multi)arrangement is free, 3-arrangements are always locally free and restricted multiarrangements are also free. So (1) and (3) hold for any 3-arrangement. But it is not necessarily free. We also note that Kung [Ku] found many examples of non-free arrangements whose characteristic polynomial factors completely over \mathbb{Z} . Recently, Schenck [Sc2] studies the difference between freeness and the factorization of characteristic polynomials for 3-arrangements.

It seems natural to ask whether a combination of some conditions characterizes freeness. The behavior is completely different for $\ell = 3$ or $\ell \geq 4$.

For 3-arrangement \mathcal{A} we will prove, in §3, that \mathcal{A} is free if and only if it satisfies (2)+(3) with the coincidence of numbers, i.e. the condition (2+3) below characterizes freeness. (Theorem 3.2)

(2+3) The characteristic polynomial factors as

$$\chi(\mathcal{A}, t) = (t - 1)(t - d_2)(t - d_3)$$

and multiexponents of restricted multiarrangement is (d_2, d_3) .

Our proof is based on a study of Solomon-Terao's formula for characteristic polynomial and Ziegler's restriction map using Hilbert series.

In §4, we will study the freeness of an arrangement from the viewpoint of coherent sheaves on projective space. The freeness of an $\ell(\geq 4)$ -arrangement can be characterized by (1) and (3). Furthermore, with the help of results on reflexive sheaves, we will give a characterization by the following weaker condition (Theorem 4.9).

- (1'+3) Let $H \in \mathcal{A}$ be a hyperplane. The restricted multiarrangement on H is free and \mathcal{A} is locally free along H .

Edelman-Reiner [ER2] conjectured that cones over certain truncated affine Weyl arrangements are free. As an application of our characterization of freeness, we prove that the Edelman-Reiner conjecture is true. The first half of condition (1'+3) has been proved by Terao [Te3]. The second half will be proved by induction on the rank of root system using following fact: any localization of Weyl arrangement decomposes into a direct sum of Weyl arrangements of lower ranks. Indeed, the required local freeness is equivalent to the Edelman-Reiner conjecture for root systems of strictly lower ranks. The problem is resolved into computation of characteristic polynomials for rank two root systems, which has been verified by Athanasiadis [Ath1, Ath3]. And we also give a family of free arrangements which interpolates between extended Shi and extended Catalan arrangements.

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2 Preliminaries

Let V be an ℓ -dimensional linear space over an arbitrary field \mathbb{K} of characteristic 0 and $S := \mathbb{K}[V^*]$ be the algebra of polynomial functions on V that is naturally isomorphic to $\mathbb{K}[z_1, z_2, \dots, z_\ell]$ for any choice of basis (z_1, \dots, z_ℓ) of V^* .

A (central) hyperplane arrangement \mathcal{A} is a finite collection of codimension one linear subspaces in V . For each hyperplane H of \mathcal{A} , fix a nonzero linear form $\alpha_H \in V^*$ vanishing on H and put $Q := \prod_{H \in \mathcal{A}} \alpha_H$.

The characteristic polynomial of \mathcal{A} is defined as

$$\chi(\mathcal{A}, t) = \sum_{X \in L_{\mathcal{A}}} \mu(X) t^{\dim X},$$

where $L_{\mathcal{A}}$ is a lattice consists of the intersections of elements of \mathcal{A} , ordered by reverse inclusion, $\hat{0} := V$ is the unique minimal element of $L_{\mathcal{A}}$ and $\mu : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ is the Möbius function defined as follows:

$$\begin{aligned} \mu(\hat{0}) &= 1, \\ \mu(X) &= - \sum_{Y < X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

Denote by Der_V and Ω_V^p , respectively, the S -module of all polynomial vector fields and of polynomial differential p -forms over V .

Definition 2.1 *For a given arrangement \mathcal{A} , we define modules of logarithmic vector fields and logarithmic p -forms by, respectively,*

$$D(\mathcal{A}) = \{\delta \in \text{Der}_V \mid \delta(\alpha_H) \in \alpha_H S, \forall H \in \mathcal{A}\}$$

and

$$\Omega_V^p(\mathcal{A}) = \left\{ \omega \in \frac{1}{Q} \Omega_V^p \mid Q \cdot \frac{d\alpha_H}{\alpha_H} \wedge \omega \in \Omega_V^{p+1}, \forall H \in \mathcal{A} \right\}.$$

Next we recall a formula due to Solomon and Terao which deduces the characteristic polynomial from the Hilbert series of the graded S -modules $\Omega^p(\mathcal{A})$. For a finitely generated graded S -module M , the series $P(M, x) \in \mathbb{Z}[x^{-1}][[x]]$ defined by

$$P(M, x) = \sum_{p \in \mathbb{Z}} (\dim_{\mathbb{K}} M_p) x^p$$

is called the Hilbert series.

For an arrangement \mathcal{A} , define

$$\Phi(\mathcal{A}; x, y) = \sum_{p=0}^{\ell} P(\Omega^p(\mathcal{A}), x) y^p.$$

Theorem 2.2 [ST1] *The characteristic polynomial $\chi(\mathcal{A}, t)$ is expressed as*

$$\chi(\mathcal{A}, t) = \lim_{x \rightarrow 1} \Phi(\mathcal{A}; x, t(1-x) - 1).$$

An arrangement \mathcal{A} is said to be **free** if $D(\mathcal{A})$ (or equivalently $\Omega^1(\mathcal{A})$) is a free S -module, and then the multiset of degrees $\exp(\mathcal{A}) := (d_1, d_2, \dots, d_{\ell})$ of a homogeneous basis of $D(\mathcal{A})$ is called the **exponents**.

Theorem 2.2 yields a famous factorization theorem by Terao;

Theorem 2.3 [Te2] *If \mathcal{A} is a free arrangement with exponents $(d_1, d_2, \dots, d_{\ell})$, the characteristic polynomial factors as follows;*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

A multiarrangement (introduced by Ziegler [Zi]) is a pair $(\mathcal{A}, \mathbf{k})$ consisting of an ordinary arrangement \mathcal{A} and a map $\mathbf{k} : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ (called multiplicity). Any arrangement can be considered as a multiarrangement with constant multiplicity $\mathbf{k}(H) = 1, \forall H \in \mathcal{A}$.

Definition 2.4 Let $(\mathcal{A}, \mathbf{k} : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0})$ be a multiarrangement. Denote $Q(\mathcal{A}, \mathbf{k}) := \prod_{H \in \mathcal{A}} \alpha_H^{\mathbf{k}(H)}$. We define the modules $D(\mathcal{A}, \mathbf{k})$ and $\Omega^p(\mathcal{A}, \mathbf{k})$ by

$$D(\mathcal{A}, \mathbf{k}) = \{\delta \in \text{Der}_V \mid \delta(\alpha_H) \in \alpha_H^{\mathbf{k}(H)} S, \forall H \in \mathcal{A}\}$$

and

$$\Omega^p(\mathcal{A}, \mathbf{k}) = \left\{ \omega \in \frac{1}{Q(\mathcal{A}, \mathbf{k})} \Omega_V^p \mid Q(\mathcal{A}, \mathbf{k}) \cdot \frac{d\alpha_H}{\alpha_H^{\mathbf{k}(H)}} \wedge \omega \in \Omega_V^{p+1}, \forall H \in \mathcal{A} \right\}.$$

The following is straightforward.

$$\Omega^0(\mathcal{A}, \mathbf{k}) = S, \quad \Omega^\ell(\mathcal{A}, \mathbf{k}) = \frac{1}{Q(\mathcal{A}, \mathbf{k})} \Omega_V^\ell.$$

A multiarrangement $(\mathcal{A}, \mathbf{k})$ is said to be free if $D(\mathcal{A}, \mathbf{k})$ (or equivalently $\Omega^1(\mathcal{A}, \mathbf{k})$) is a free S -module. In this case, the multiset of degrees of a homogeneous basis of $D(\mathcal{A}, \mathbf{k})$ is called the **multiexponents** and also denoted by $\exp(\mathcal{A}, \mathbf{k})$. We list some consequences of the freeness of a multiarrangements that will be used later.

Theorem 2.5 [Sa1](Saito's criterion) Let $\omega_1, \dots, \omega_\ell \in \Omega^1(\mathcal{A}, \mathbf{k})$ be homogeneous and linearly independent over S . Then $(\mathcal{A}, \mathbf{k})$ is free with basis $\omega_1, \dots, \omega_\ell$ if and only if

$$\sum_{i=1}^{\ell} \deg \omega_i = - \sum_{H \in \mathcal{A}} \mathbf{k}(H).$$

Theorem 2.6 (Localization Theorem) Let $(\mathcal{A}, \mathbf{k})$ be a free multiarrangement. For any intersection $X \in L_{\mathcal{A}}$, define

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid H \ni X\}.$$

Then the induced multiarrangement $(\mathcal{A}_X, \mathbf{k}|_{\mathcal{A}_X})$ is also free.

Example 2.7 Let \mathcal{A} be an arrangement in V and $H_1 \in \mathcal{A}$ be a hyperplane. Then the restriction of \mathcal{A} to H_1 is the arrangement $\mathcal{A}^{H_1} := \{H \cap H_1 \mid H \in \mathcal{A} \setminus \{H_1\}\}$ in H_1 . This restriction has a natural structure of a multiarrangement $(\mathcal{A}^{H_1}, \mathbf{k}_{\mathcal{A}}^{H_1})$ with multiplicity $\mathbf{k}_{\mathcal{A}}^{H_1} : \mathcal{A}^{H_1} \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\mathbf{k}_{\mathcal{A}}^{H_1} : \mathcal{A}^{H_1} \ni H' \longmapsto \sharp(\{H \in \mathcal{A} \mid H \cap H_1 = H'\}).$$

Fix a coordinate system $(z_1, z_2, \dots, z_\ell)$ such that $H_1 = \{z_1 = 0\}$. Then every $\omega \in \Omega^p(\mathcal{A})$ can be uniquely expressed as

$$\omega = \omega_1 + \frac{dz_1}{z_1} \wedge \omega_2,$$

where ω_1 and ω_2 are rational differential forms in dz_2, \dots, dz_ℓ . The following theorem was proved by Ziegler [Zi]. (For another formulation by using the modules of vector fields, see Theorem 4.1.)

Theorem 2.8 Using the notation above, $\omega_1|_{H_1}$ is contained in $\Omega^p(\mathcal{A}^{H_1}, \mathbf{k})$. Furthermore, \mathcal{A} is free if and only if the restricted multiarrangement $(\mathcal{A}^{H_1}, \mathbf{k})$ is free and the restriction map (for $p = 1$)

$$\Omega^1(\mathcal{A}) \longrightarrow \Omega^1(\mathcal{A}^{H_1}, \mathbf{k})$$

$$\omega = \omega_1 + \frac{dz_1}{z_1} \wedge \omega_2 \longmapsto \omega_1|_{H_1}$$

is surjective.

Let us denote by

$$\mathbf{res}_H^p : \Omega^p(\mathcal{A}) \longrightarrow \Omega^p(\mathcal{A}^H, \mathbf{k}) \left(\omega_1 + \frac{dz_1}{z_1} \wedge \omega_2 \mapsto \omega_1|_H \right)$$

the restriction map and by $M^p \subset \Omega^p(\mathcal{A}^H, \mathbf{k})$ the image of \mathbf{res}_H^p . M^p is a graded $\mathbb{K}[H] := S/z_1 S$ submodule by definition. Though the next corollary is easily deduced from Theorem 2.8 and Saito's criterion, it is a starting point of our characterization of freeness.

Corollary 2.9 If the restriction map $\mathbf{res}_H^1 : \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}^H, \mathbf{k})$ is surjective and the restricted multiarrangement $(\mathcal{A}^H, \mathbf{k})$ is free with multiexponents (d'_2, \dots, d'_ℓ) , then \mathcal{A} is free with exponents $(1, d'_2, \dots, d'_\ell)$.

Proof. Surjectivity implies that there exist $\omega_2, \dots, \omega_\ell \in \Omega^1(\mathcal{A})$ such that $\text{res}_H^1(\omega_2), \dots, \text{res}_H^1(\omega_\ell) \in \Omega^p(\mathcal{A}^H, \mathbf{k})$ form a basis. Then from Saito's criterion 2.5, $\omega_2, \dots, \omega_\ell$ together with $d\alpha/\alpha \in \Omega^1(\mathcal{A})$ form a basis of $\Omega^1(\mathcal{A})$. \square

Since $\Omega^\bullet(\mathcal{A})$ is closed under exterior product, we have a homomorphism (where $\alpha = \alpha_H$ is a defining equation of $H \in \mathcal{A}$)

$$\partial : \Omega^p(\mathcal{A}) \longrightarrow \Omega^{p+1}(\mathcal{A}), \quad (\omega \longmapsto (d\alpha/\alpha) \wedge \omega).$$

Propositon 2.10 ([OT, Prop. 4.86]) *The complex $(\Omega^\bullet(\mathcal{A}), \partial)$ is acyclic.*

Using this proposition, we study the effect of restriction map on the Hilbert series.

Theorem 2.11 *The Hilbert series of M^p and $\Omega^p(\mathcal{A})$ are connected by the relationship*

$$\sum_{p=0}^{\ell-1} P(M^p, x) y^p = \frac{x(1-x)}{x+y} \times \Phi(\mathcal{A}; x, y). \quad (1)$$

Proof. The restriction map above can be considered as a composition of $\frac{dz_1}{z_1}$ and the Poincaré residue map,

$$\begin{array}{ccc} \Omega^p(\mathcal{A}) & \xrightarrow{\frac{dz_1}{z_1}} & \frac{dz_1}{z_1} \wedge \Omega^p(\mathcal{A}) \\ \text{res}_H^p \downarrow & \swarrow \text{Residue map} & \\ M^p & & \end{array}.$$

First we consider the Hilbert series of the submodules $(dz_1/z_1) \wedge \Omega^{p-1}(\mathcal{A}) \subset \Omega^p(\mathcal{A})$. The following short exact sequence is obtained from Prop 2.10,

$$0 \longrightarrow \frac{dz_1}{z_1} \wedge \Omega^{p-1}(\mathcal{A}) \longrightarrow \Omega^p(\mathcal{A}) \xrightarrow{\frac{dz_1}{z_1} \wedge} \frac{dz_1}{z_1} \wedge \Omega^p(\mathcal{A}) \longrightarrow 0.$$

We have a formula on Hilbert series ($p = 0, 1, \dots, \ell$)

$$P(\Omega^p(\mathcal{A}), x) = P\left(\frac{dz_1}{z_1} \wedge \Omega^{p-1}(\mathcal{A}), x\right) + x \cdot P\left(\frac{dz_1}{z_1} \wedge \Omega^p(\mathcal{A}), x\right).$$

Summing up with multiply y^p , we have

$$\Phi(\mathcal{A}; x, y) = \left(1 + \frac{x}{y}\right) \times \sum_{p=1}^{\ell} P\left(\frac{dz_1}{z_1} \wedge \Omega^{p-1}(\mathcal{A}), x\right) y^p \quad (2)$$

Next we consider the residue map. For general graded S -module M , suppose $s \in S$ (with $\deg = 1$) is an M -regular element, i.e. $s \cdot : M \xrightarrow{s} M$ is injective, then from the exact sequence $0 \rightarrow sM \rightarrow M \rightarrow M/sM \rightarrow 0$, we can compute the Hilbert series of M/sM as

$$P(M/sM, x) = (1 - x)P(M, x). \quad (3)$$

Using (3), we can relate the Hilbert series of the module $M^p \subset \Omega^p(\mathcal{A}^{H_1}, \mathbf{k})$ with that of $\Omega^p(\mathcal{A})$.

$$P(M^p, x) = x(1 - x)P\left(\frac{dz_1}{z_1} \wedge \Omega^p(\mathcal{A}), x\right). \quad (4)$$

Combining (2) and (4), we have the desired result. \square

3 Free arrangements in \mathbb{K}^3

In this section, let \mathcal{A} be a central arrangement in $V = \mathbb{K}^3$.

Given a hyperplane $H \in \mathcal{A}$, we have considered the natural multiarrangement $(\mathcal{A}^H, \mathbf{k})$ on H . Since $\Omega^1(\mathcal{A}^H, \mathbf{k})$ is reflexive $\mathbb{K}[H]$ -module and $\dim \mathbb{K}[H] = 2$, $\Omega^1(\mathcal{A}^H, \mathbf{k})$ is a free $\mathbb{K}[H]$ -module, we let (d'_2, d'_3) denote the multiexponents. Since the sum of multiplicities of the multiarrangement $(\mathcal{A}^H, \mathbf{k})$ is $\sharp(\mathcal{A}) - 1$, from Saito's criterion 2.5, we have

$$d'_2 + d'_3 = \sharp(\mathcal{A}) - 1. \quad (5)$$

We call $\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t-1)$ the **reduced characteristic polynomial**. If \mathcal{A} is a free arrangement, it follows from results of Ziegler (Theorem 2.8) that the exponents of \mathcal{A} are $\exp(\mathcal{A}) = (1, d'_2, d'_3)$, and by Terao (Theorem 2.3) that

$$\chi_0(\mathcal{A}, t) = (t - d'_2)(t - d'_3). \quad (6)$$

Conversely, if the multiexponent $\exp(\mathcal{A}, \mathbf{k}) = (d'_2, d'_3)$ are not roots of the characteristic polynomial $\chi(\mathcal{A}, t)$, \mathcal{A} cannot be free even if $\chi(\mathcal{A}, t)$ factors into linear terms over \mathbb{Z} .

Example 3.1 (Stanley's example) *Let \mathcal{A} be the cone of the affine 2-arrangement in Fig.1(left). Then the characteristic polynomial factors as $\chi(\mathcal{A}, t) = (t - 1)(t - 3)^2$. However, the restriction of \mathcal{A} to the hyperplane at infinity, Fig.1(right), has exponents $(1, 5)$, hence \mathcal{A} is not free.*

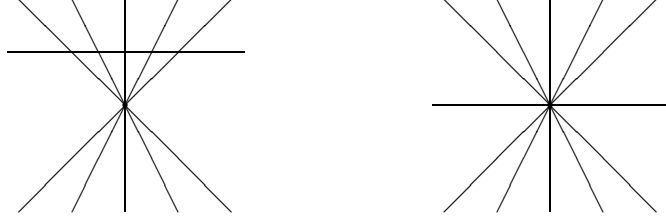


Figure 1: Stanley's example and its restriction to the hyperplane at infinity.

Because of relation (5), the equation (6) is equivalent to $\chi_0(\mathcal{A}, 0) = d'_2 \cdot d'_3$. The main result of this section is that these relations characterize the freeness of \mathcal{A} .

Theorem 3.2 *Let \mathcal{A} be an arrangement in \mathbb{K}^3 , $\chi_0(\mathcal{A}, t) := (1 - t)^{-1} \chi(\mathcal{A}, t)$ be the reduced characteristic polynomial, (d'_2, d'_3) be the multiexponents of restricted multiarrangement $(\mathcal{A}^H, \mathbf{k})$ and M^1 be the image of restriction map $\text{res}_H^1 : \Omega^1(\mathcal{A}) \longrightarrow \Omega^1(\mathcal{A}^H, \mathbf{k})$. Then the codimension of M^1 in $\Omega^1(\mathcal{A}^H, \mathbf{k})$ is finite and is given by*

$$\chi_0(\mathcal{A}, 0) - d'_2 \cdot d'_3.$$

In particular, from Corollary 2.9,

Corollary 3.3 *\mathcal{A} is free if and only if $\chi(\mathcal{A}, t) = (t - 1)(t - d'_2)(t - d'_3)$, where (d'_2, d'_3) are the multiexponents of the restricted multiarrangement.*

Proof of Theorem. From the assumption, $\Omega^1(\mathcal{A}^H, \mathbf{k})$ has a homogeneous basis with degrees $(-d'_2, -d'_3)$. Hence it follows from the simple computation that

$$\lim_{x \rightarrow 1} \sum_{p=0}^2 P(\Omega^p(\mathcal{A}^H, \mathbf{k}), x) (t(1 - x) - 1)^p = (t - d'_2)(t - d'_3). \quad (7)$$

Recall the characteristic polynomial can be calculated from

$$\Phi(\mathcal{A} : x, y) = \sum_{p=0}^3 P(\Omega^p(\mathcal{A}), x) y^p$$

by Theorem 2.2. We compare the Hilbert series above to that of $\Omega^p(\mathcal{A}^H, \mathbf{k})$ ($p = 0, 1, 2$). From Theorem 2.11, $\Phi(\mathcal{A}; x, y)$ can be expressed by the Hilbert series of M^p

$$\Phi(\mathcal{A}; x, y) = \frac{x + y}{x(1 - x)} \sum_{p=0}^2 P(M^p, x) y^p.$$

Hence,

$$\begin{aligned} \chi_0(\mathcal{A}, t) &= \frac{1}{t - 1} \lim_{x \rightarrow 1} \frac{x + t(1 - x) - 1}{x(1 - x)} \sum_{p=0}^2 P(M^p, x) (t(1 - x) - 1)^p \\ &= \lim_{x \rightarrow 1} \sum_{p=0}^2 P(M^p, x) (t(1 - x) - 1)^p. \end{aligned}$$

We note that the maps $\mathbf{res}_H^0, \mathbf{res}_H^2$ are naturally surjective. Hence, (recall (5))

$$\begin{aligned} P(M^0, x) &= P(\Omega^0(\mathcal{A}^H, \mathbf{k}), x) = \frac{1}{(1 - x)^2} \\ P(M^2, x) &= P(\Omega^2(\mathcal{A}^H, \mathbf{k}), x) = \frac{x^{1-\sharp(\mathcal{A})}}{(1 - x)^2}, \end{aligned}$$

and we have from the definition,

$$\dim_{\mathbb{K}} (\Omega^1(\mathcal{A}^H, \mathbf{k})/M^1) = \lim_{x \rightarrow 1} [P(\Omega^1(\mathcal{A}^H, \mathbf{k}), x) - P(M^1, x)] \quad (8)$$

Using above relations,

$$\begin{aligned} \chi_0(\mathcal{A}, 0) - d'_1 \cdot d'_2 &= \chi_0(\mathcal{A}, t) - (t - d'_1)(t - d'_2) \\ &= \lim_{x \rightarrow 1} (t(1 - x) - 1) [P(M^1, x) - P(\Omega^1(\mathcal{A}^H, \mathbf{k}), x)] \\ &= t \cdot \lim_{x \rightarrow 1} (1 - x) [P(M^1, x) - P(\Omega^1(\mathcal{A}^H, \mathbf{k}), x)] \\ &\quad - \lim_{x \rightarrow 1} [P(M^1, x) - P(\Omega^1(\mathcal{A}^H, \mathbf{k}), x)] \end{aligned}$$

Since left hand side is a constant, we have

$$\dim_{\mathbb{K}} (\Omega^1(\mathcal{A}^H, \mathbf{k})/M^1) = \chi_0(\mathcal{A}, 0) - d'_1 \cdot d'_2.$$

□

4 Free arrangements in $\mathbb{K}^{\ell+1}$ ($\ell \geq 3$)

Let \mathcal{A} be an essential arrangement in $V = \mathbb{K}^{\ell+1}$ ($\ell \geq 3$). Fix a hyperplane H_0 and coordinate system $(z_0, z_1, \dots, z_\ell)$ such that $H_0 = \{z_0 = 0\}$. Denote by $S := \mathbb{K}[z_0, z_1, \dots, z_\ell]$ as usual.

Define S -submodules $D_0(\mathcal{A})$ and $\Omega_0^1(\mathcal{A})$ of $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$, respectively, by

$$\begin{aligned} D_0(\mathcal{A}) &:= \{\delta \in D(\mathcal{A}) \mid \delta z_0 = 0\} \\ \Omega_0^1(\mathcal{A}) &:= \{\omega \in \Omega^1(\mathcal{A}) \mid \langle E, \omega \rangle = 0\}, \end{aligned}$$

where E is the Euler vector field and \langle, \rangle is the inner product. We have the following splitting as S -modules:

$$\begin{aligned} D(\mathcal{A}) &= S \cdot E \oplus D_0(\mathcal{A}) \\ \Omega^1(\mathcal{A}) &= S \cdot \frac{dz_0}{z_0} \oplus \Omega_0^1(\mathcal{A}). \end{aligned}$$

The duality between $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ implies that the modules $D_0(\mathcal{A})$ and $\Omega_0^1(\mathcal{A})$ are dual each other, hence they are reflexive. Contrary to previous sections, we consider the module of vector fields $D(\mathcal{A})$ (or $D_0(\mathcal{A})$) in this section. Ziegler's restriction map can be formulated as follows.

$$\begin{aligned} D_0(\mathcal{A}) &\longrightarrow D(\mathcal{A}^{H_1}, \mathbf{k}) \\ \delta &\longmapsto \delta|_{z_0=0}. \end{aligned} \tag{9}$$

So we have an exact sequence,

$$0 \longrightarrow D_0(\mathcal{A}) \xrightarrow{z_0 \cdot} D_0(\mathcal{A}) \longrightarrow D(\mathcal{A}^{H_0}, \mathbf{k}). \tag{10}$$

Theorem 2.8 and Corollary 2.9 can be also stated as follows:

Theorem 4.1 *\mathcal{A} is free with $\exp(\mathcal{A}) = (e_0 (= 1), e_1, \dots, e_\ell)$ if and only if $D(\mathcal{A}^{H_0}, \mathbf{k})$ is free with exponents (e_1, \dots, e_ℓ) and the restriction map (9) is surjective for some $H_0 \in \mathcal{A}$.*

From now on we consider reflexive $\mathcal{O}_{\mathbb{P}^\ell}$ -module $\widetilde{D_0(\mathcal{A})}$ on \mathbb{P}^ℓ rather than the graded S -module $D_0(\mathcal{A})$ [Ha1]. The local structure of the coherent sheaf

$\widetilde{D_0(\mathcal{A})}$ can be described by the local structure of arrangement \mathcal{A} . More precisely, if we define the localization of \mathcal{A} at x by

$$\mathcal{A}_x := \{H \in \mathcal{A} \mid H \ni x\}$$

for $x \in V$, the stalk $\widetilde{D(\mathcal{A})}_{\bar{x}}$ at $\bar{x} \in \mathbb{P}^\ell$ is isomorphic to that of $\widetilde{D(\mathcal{A}_x)}$.

$$\widetilde{D(\mathcal{A})}_{\bar{x}} \cong \widetilde{D(\mathcal{A}_x)_{\bar{x}}}. \quad (11)$$

In particular, $\widetilde{D(\mathcal{A})}$ is a locally free sheaf on \mathbb{P}^ℓ if and only if \mathcal{A} is locally free, i.e. \mathcal{A}_X is free for all $X \in L_{\mathcal{A}} \setminus \{0\}$. (For details see Mustață and Schenck [MS, Thm 2.3])

From the discussion above, we have an exact sequence

$$0 \longrightarrow \widetilde{D_0(\mathcal{A})}(d-1) \longrightarrow \widetilde{D_0(\mathcal{A})}(d) \longrightarrow D(\widetilde{\mathcal{A}^{H_0}}, \mathbf{k})(d)$$

($d \in \mathbb{Z}$). But the last homomorphism is not necessarily surjective. For the sake of surjectivity we consider the following condition.

Condition 4.2 *Arrangement \mathcal{A} is locally free along H_0 , i.e. \mathcal{A}_x is free for all $x \in H_0 \setminus \{0\}$.*

Note that locally free arrangements satisfy this condition.

Propositon 4.3 *If Condition 4.2 holds then the restriction map induces an exact sequence,*

$$0 \longrightarrow \widetilde{D_0(\mathcal{A})}(d-1) \longrightarrow \widetilde{D_0(\mathcal{A})}(d) \longrightarrow D(\widetilde{\mathcal{A}^{H_0}}, \mathbf{k})(d) \longrightarrow 0, \quad (12)$$

and we have $\widetilde{D_0(\mathcal{A})}(d)|_{\mathbb{P}(H_0)} = D(\widetilde{\mathcal{A}^{H_0}}, \mathbf{k})(d)$, where $\mathbb{P}(H_0)$ is the projective hyperplane defined by $H_0 \subset V$.

Proof. What we have to show is the surjectivity. Since \mathcal{A}_x is free for all $x \in H_0 \setminus \{0\}$, the induced restriction map

$$D_0(\mathcal{A}_x) \longrightarrow D(\mathcal{A}_x^{H_0}, \mathbf{k}|_{\mathcal{A}_x^{H_0}})$$

is surjective by Theorem 4.1. From (11), this shows the surjectivity of the homomorphism as sheaves. \square

In the context of coherent sheaves on projective space, an arrangement \mathcal{A} is free with $\exp(\mathcal{A}) = (e_0 (= 1), e_1, \dots, e_\ell)$ if and only if the coherent sheaf $\widetilde{D_0(\mathcal{A})}$ splits into a direct sum of line bundles as

$$\widetilde{D_0(\mathcal{A})} = \mathcal{O}_{\mathbb{P}^\ell}(-e_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^\ell}(-e_\ell).$$

In the theory of vector bundles on projective space, the following remarkable theorem is known.

Theorem 4.4 ([OSS, Theorem 2.3.2.])

A holomorphic vector bundle \mathcal{E} on \mathbb{P}^ℓ splits into a direct sum of line bundles precisely when its restriction to some plane $\mathbb{P}^2 \subset \mathbb{P}^\ell$ splits.

In particular, in the case $\ell \geq 3$, a vector bundle \mathcal{E} on \mathbb{P}^ℓ splits if its restriction to some hyperplane $\mathbb{P}^{\ell-1}$ splits. Here we consider the following condition.

Condition 4.5 *The restricted multiarrangement $(\mathcal{A}^{H_0}, \mathbf{k}_{\mathcal{A}}^{H_0})$ is free.*

From the discussion above, we conclude the following result.

Corollary 4.6 *Let \mathcal{A} be an arrangement in $\mathbb{K}^{\ell+1}$ ($\ell \geq 3$). \mathcal{A} is free if and only if \mathcal{A} is locally free and the restricted multiarrangement is free (=Condition 4.5) for some hyperplane of \mathcal{A} .*

The aim of this section is to characterize freeness by Condition 4.2. This condition is weaker than local freeness. If \mathcal{A} is not a locally free arrangement, $\widetilde{D_0(\mathcal{A})}$ is not a vector bundle. In this case the proof of Theorem 4.4 does not work for $\widetilde{D_0(\mathcal{A})}$. We recall some results on reflexive sheaves over projective spaces. For details see [Ha2].

A coherent sheaf \mathcal{F} on a scheme X is said to be reflexive if the natural map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ of \mathcal{F} to its double dual is an isomorphism. If X is a regular scheme, i.e. all its local rings are regular local rings, then a reflexive sheaf \mathcal{F} on X is locally free except along a closed subscheme Y of codimension ≥ 3 ([Ha2, Corollary 1.4.]).

Now we prove

Lemma 4.7 *Let \mathcal{F} be a reflexive sheaf on \mathbb{P}^n ($n \geq 3$) and assume that \mathcal{F} is locally free except at a finite number of points $\{P_i\}$. Then there exist a surjection*

$$H^{n-1}(\mathcal{F}^\vee \otimes \omega) \longrightarrow \text{Ext}^{n-1}(\mathcal{F}, \omega) \longrightarrow 0,$$

where $\omega = \mathcal{O}(-n-1)$ is the dualizing sheaf on \mathbb{P}^n .

Proof. (See also [Ha2, Theorem 2.4.]) Consider the spectral sequence of local and global Ext functors:

$$E_2^{p,q} = H^p(\mathcal{E}xt^q(\mathcal{F}, \omega)) \implies E^{p+q} = \text{Ext}^{p+q}(\mathcal{F}, \omega).$$

From the assumption, the non-zero $E_2^{p,q}$ terms appear either $p = 0$ or $q = 0$. Indeed, since \mathcal{F} is locally free except for $\{P_i\}$, supports of $\mathcal{E}xt^q(\mathcal{F}, \omega)$ ($q \geq 1$) are contained in finite set $\{P_i\}$. It follows that $E_2^{p,q} = 0$ for $p, q \geq 1$. Furthermore, at these points \mathcal{F} has depth ≥ 2 ([Ha2, Prop. 1.3.]), hence has homological dimension $\leq n - 2$. Thus $\mathcal{E}xt^q(\mathcal{F}, \omega) = 0$ for $q \geq n - 1$. Now the result is straightfoward from the definition of spectral sequence. \square

Since $\text{Ext}^{n-1}(\mathcal{F}(d), \omega)$ is Serre dual to $H^1(\mathbb{P}^n, \mathcal{F}(d))$, we have the inequality

$$\dim H^1(\mathcal{F}(d)) \leq \dim H^{n-1}(\mathcal{F}(d)^\vee \otimes \omega),$$

by the theorem. The right-hand side vanishes for $d \ll 0$.

Corollary 4.8 $H^1(\mathbb{P}^n, \mathcal{F}(d)) = 0$ for $d \ll 0$.

We now prove the main theorem of this section.

Theorem 4.9 *Let \mathcal{A} be a arrangement in $\mathbb{K}^{\ell+1}$ ($\ell \geq 3$) and fix a hyperplane $H \in \mathcal{A}$. Then \mathcal{A} is free if (and only if) \mathcal{A} is locally free along H (=Condition 4.2) and the restricted multiarrangement $(\mathcal{A}^H, \mathbf{k}_{\mathcal{A}}^H)$ is free (=Condition 4.5).*

Proof. We first note that $\widetilde{D_0(\mathcal{A})}$ is locally free except for finite points. Indeed, if there exists an $X \in L_{\mathcal{A}}$ with $\dim X \geq 2$ such that \mathcal{A}_X is not free, X must intersect with the hyperplane H . Then $X \cap H$ is a set at which $\widetilde{D_0(\mathcal{A})}$ is not locally free, which contradicts the assumption that \mathcal{A} is locally free along H .

From the vanishing of intermediate cohomology groups of line bundles over projective space, we have

$$H^i(\mathbb{P}(H), \widetilde{D_0(\mathcal{A})}|_{\mathbb{P}(H)}) = 0, \text{ for } 1 \leq i \leq \ell - 2.$$

Hence the next exact sequence is obtained ($\mathcal{F} := \widetilde{D_0(\mathcal{A})}$).

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{F}(d-1)) \longrightarrow H^0(\mathcal{F}(d)) \longrightarrow H^0(\mathcal{F}(d)|_{\mathbb{P}(H)}) \\ &\longrightarrow H^1(\mathcal{F}(d-1)) \longrightarrow H^1(\mathcal{F}(d)) \longrightarrow 0. \end{aligned}$$

The surjection in the second row and Corollary 4.8 indicate that $H^1(\mathcal{F}(d)) = 0$ for any $d \in \mathbb{Z}$. Thus $H^0(\mathcal{F}(d)) \rightarrow H^0(\mathcal{F}(d)|_{\mathbb{P}(H)})$ is surjective for any $d \in \mathbb{Z}$. This implies that

$$D_0(\mathcal{A}) \rightarrow D(\mathcal{A}^H, \mathbf{k}_{\mathcal{A}}^H) \rightarrow 0$$

is surjective. From (2.9) we conclude that \mathcal{A} is free. \square

5 Application

We use Theorem 4.9 to show that cones over a certain truncated affine Weyl arrangements are free.

Let $V = \mathbb{R}^\ell$ be an ℓ -dimensional Euclidean space with a coordinate system (x_1, \dots, x_ℓ) . Let Φ be a crystallographic irreducible root system in V with exponents $(e_1, e_2, \dots, e_\ell)$ and Coxeter number h . We also fix a positive root system $\Phi^+ \subset \Phi$. For each positive root $\alpha \in \Phi^+$ and integer $k \in \mathbb{Z}$, define an affine hyperplane $H_{\alpha,k}$ by

$$H_{\alpha,k} := \{v \in V \mid \alpha(v) = k\}.$$

We have a hyperplane arrangement

$$\mathcal{A}(\Phi^+) := \{H_{\alpha,0} \mid \alpha \in \Phi^+\}$$

in V , called the Weyl arrangement associated to Φ . The Weyl arrangement is free, more generally,

Theorem 5.1 [Sa2] *Let \mathcal{A} be a Coxeter arrangement, i.e. the set of all reflecting hyperplanes of a finite Coxeter group of exponents (e_1, \dots, e_ℓ) . Then \mathcal{A} is a free arrangement with $\exp(\mathcal{A}) = (e_1, \dots, e_\ell)$.*

The basis of $D(\mathcal{A})$ can be constructed explicitly in terms of the invariant theory of Coxeter groups. We next define a family of arrangements in $\mathbb{R} \times V$ (with coordinate system $(x_0, x_1, \dots, x_\ell)$) associated with an affine Weyl arrangement. These kinds of arrangements were first studied by Shi [Sh1, Sh2].

Definition 5.2 *For integers $p, q \in \mathbb{Z}$ with $p \leq q$, denote by $[p, q]$ the set $\{p, p+1, \dots, q\}$ of integers from p to q . We define an affine arrangement in V as follows:*

$$\mathcal{A}(\Phi^+)^{[p,q]} := \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in [p, q]\},$$

and its cone in $\mathbb{R} \times V$

$$\mathbf{c}\mathcal{A}(\Phi^+)^{[p,q]} := \{\alpha - kx_0 \mid \alpha \in \Phi^+, k \in [p, q]\} \cup \{H_\infty := \{x_0 = 0\}\}.$$

Edelman and Reiner [ER2] posed a conjecture on the freeness for such kind of arrangements.

Conjecture 5.3 [ER2]

- (1) The cone $\mathbf{c}\mathcal{A}(\Phi^+)^{[-m,m]}$ ($m \geq 0$) of an extended Catalan arrangement $\mathcal{A}(\Phi^+)^{[-m,m]}$, ($m \geq 0$) is free with exponents $(1, e_1 + mh, e_2 + mh, \dots, e_\ell + mh)$.
- (2) The cone $\mathbf{c}\mathcal{A}(\Phi^+)^{[1-m,m]}$ ($m \geq 1$) of an extended Shi arrangement $\mathcal{A}(\Phi^+)^{[1-m,m]}$, ($m \geq 1$) is free with exponents $(1, mh, mh, \dots, mh)$.

By the general theory of free arrangements, we can deduce some conclusions from the conjecture. First restricting to the hyperplane at infinity $H_\infty = \{x_0 = 0\}$, from Theorem 2.8, we have.

Conclusion 5.4 (Multi-freeness) For $\mathcal{A} = \mathcal{A}(\Phi^+)$ and $n \geq 0$,

$$D(\mathcal{A}, n) := \{\delta \in \text{Der}_V \mid \delta\alpha_H \in (\alpha_H^n), \forall H \in \mathcal{A}\}$$

is free with multiexponents

$$\begin{cases} (e_1 + mh, \dots, e_\ell + mh), & \text{if } n = 2m + 1, \\ (mh, \dots, mh), & \text{if } n = 2m. \end{cases}$$

Second, from the factorization theorem 2.3, we have another conclusion.

Conclusion 5.5 (Factorization) The characteristic polynomial of these arrangements are given by

$$\begin{aligned} \chi(\mathcal{A}(\Phi^+)^{[-m,m]}, t) &= \prod_{i=1}^{\ell} (t - e_i - mh) \\ \chi(\mathcal{A}(\Phi^+)^{[1-m,m]}, t) &= (t - mh)^\ell. \end{aligned}$$

In the case of root system of type A , Conjecture 5.3 (1) and (2) have been proved by Edelman and Reiner [ER2] and Athanasiadis [Ath2], respectively.

Without assuming conjecture, conclusion 5.4 was first studied by Solomon and Terao [ST2]. Terao [Te3] has completed the proof of (5.4) for Coxeter arrangements. A generalized version is also proved [Yo].

Theorem 5.6 *Let \mathcal{A} be a Coxeter arrangement with Coxeter number h . Suppose $\mathcal{A}' \subset \mathcal{A}$ is a free subarrangement with $\exp(\mathcal{A}') = (e'_1, \dots, e'_\ell)$. Let \mathbf{k} be a multiplicity on \mathcal{A} defined by*

$$\mathbf{k}(H) = \begin{cases} 2m+1, & H \in \mathcal{A}' \\ 2m, & H \in \mathcal{A} \setminus \mathcal{A}'. \end{cases}$$

Then the multiarrangement $(\mathcal{A}, \mathbf{k})$ is free with exponents

$$\exp(\mathcal{A}, \mathbf{k}) = (e'_1 + mh, \dots, e'_\ell + mh)$$

Conclusion 5.5 has been checked by computing characteristic polynomials when Φ is of classical type, that is of type A , B , C or D , by Athanasiadis, see [Ath3]. Recently Athanasiadis [Ath4] gives a case-free proof of the equation

$$\chi(\mathcal{A}(\Phi^+)^{[-m, m]}, t) = \chi(\mathcal{A}(\Phi^+), t - mh)$$

which verifies 5.5 for extended Catalan arrangement. However, what we will need in our proof is a very weak version: (see also Prop.5.11 below)

$$\text{“Conclusion 5.5 is true for } A_2, B_2 \text{ and } G_2\text{.”} \tag{13}$$

Now let us prove Conjecture 5.3. Our proof relies on the following elementary fact on root systems.

Lemma 5.7 *Let Φ be an irreducible root system in V . Then for any point $x \in V \setminus \{0\}$, the localization of Φ at $x \in V$*

$$\Phi_x := \{ \alpha \in \Phi^+ \mid \alpha(x) = 0 \}$$

decomposes into a direct sum of root systems of lower ranks.

Let $\Phi_x = \Phi_1 \oplus \dots \oplus \Phi_k$ be an irreducible decomposition, a positive system Φ^+ naturally determines positive systems on direct summands, by $\Phi_i^+ := \Phi^+ \cap \Phi_i$.

Theorem 5.8 *Conjecture 5.3 is true for any irreducible root system.*

Proof. We prove by induction on the rank of the root system Φ . In the case of rank two, the cone $\mathbf{c}\mathcal{A}(\Phi^+)^{[p, q]}$ is a 3-arrangement. Note that rank two root system is A_2 , B_2 or G_2 . (13) and Theorem 5.6 verifies the assumptions of Corollary 3.3. Thus the assertion is true for rank two root systems.

Let Φ be an irreducible root system of higher rank. We apply Theorem 4.9 to prove freeness of $\mathbf{c}\mathcal{A}(\Phi^+)^{[p,q]}$, where $[p, q]$ is either $[1 - m, m]$ or $[-m, m]$. The freeness of the restricted multiarrangement is verified by Theorem 5.6. So what we have to check is that $\mathbf{c}\mathcal{A}(\Phi^+)^{[p,q]}$ is locally free along H_∞ .

H_∞ can be identified with $V \cong \{0\} \times V$. For given point $x \in V \setminus \{0\}$, put $\Phi_x = \Phi_1 \oplus \cdots \oplus \Phi_k$, where Φ_i are irreducible root systems which have strictly lower rank than that of Φ . Then it is easily seen that the localization of $\mathbf{c}\mathcal{A}(\Phi^+)^{[p,q]}$ at $x \in H_\infty \setminus \{0\}$ is

$$(\mathbf{c}\mathcal{A}(\Phi^+)^{[p,q]})_x = \mathbf{c}(\mathcal{A}(\Phi_1^+)^{[p,q]} \oplus \cdots \oplus \mathcal{A}(\Phi_k^+)^{[p,q]}).$$

From the inductive assumption, $\mathbf{c}\mathcal{A}(\Phi_i^+)^{[p,q]}$ is free for each $i = 1, \dots, k$. To finish the proof, we apply the next lemma. \square

Lemma 5.9 *Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be affine arrangements such that each cone $\mathbf{c}\mathcal{A}_i$ is free ($i = 1, \dots, k$). Then $\mathbf{c}(\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_k)$ is also free.*

As a possible generalization of Edelman-Reiner conjecture, we give a family of free arrangements interpolating between extended Shi $\mathbf{c}\mathcal{A}(\Phi^+)^{[1-m, m]}$ and Catalan $\mathbf{c}\mathcal{A}(\Phi^+)^{[-m, m]}$ arrangements. Recall that for positive roots $\alpha, \beta \in \Phi^+$, we denote $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. Let $\Psi \subset \Phi^+$ be a subset of positive roots satisfying following conditions (1) and (2).

- (1) $\Psi \subset \Phi^+$ is an order ideal, i.e. $\alpha \in \Psi$ and $\beta \leq \alpha \implies \beta \in \Psi$.
- (2) $\mathcal{A}(\Psi) := \{H_{\alpha, 0} \mid \alpha \in \Psi\}$ is a free arrangement (letting $\exp(\mathcal{A}(\Psi)) = (e'_1, e'_2, \dots, e'_\ell)$).

Remark 5.10 We do not know if (1) implies (2). When Φ is of type A_{n-1} , a subarrangement of $\mathcal{A}(\Phi)$ corresponds to a graph with n vertices, which is called a graphic arrangements. In this case, by using Stanley's characterization of freeness of graphic arrangement [Sta1] (see also [ER1]), any subarrangement $\mathcal{A}(\Psi)$ determined by an order ideal $\Psi \subset \Phi^+$ is free.

For $m \in \mathbb{Z}_{\geq 0}$, let us take a union of the extended Shi $\mathcal{A}(\Phi^+)^{[1-m, m]}$ and Ψ ,

$$\mathcal{A}(\Phi^+, \Psi, m) := \mathcal{A}(\Phi^+)^{[1-m, m]} \cup \{H_{\alpha, -m} \mid \alpha \in \Psi\}.$$

For our generalization, (13) should be generalized as follows, which can be proved by elementary computations.

Propositon 5.11 *Let Φ_0 be a root system of rank two, $\Psi_0 \subset \Phi_0^+$ an order ideal and h the Coxeter number. Denote by $(e'_1, e'_2) = (1, \sharp(\Psi_0) - 1)$ the exponents of $\mathcal{A}(\Psi_0)$. Then the characteristic polynomial of $\mathcal{A}(\Phi_0^+, \Psi_0, m)$ is*

$$\chi(\mathcal{A}(\Phi_0^+, \Psi_0, m), t) = (t - e'_1 - mh)(t - e'_2 - mh).$$

The proof of the next theorem is so similar to that of Theorem 5.8 that it will be omitted.

Theorem 5.12 *With notation as above, the cone $\mathbf{c}\mathcal{A}(\Phi^+, \Psi, m)$ is free with exponents*

$$\exp(\mathbf{c}\mathcal{A}(\Phi^+, \Psi, m)) = (1, e'_1 + mh, e'_2 + mh, \dots, e'_\ell + mh).$$

Corollary 5.13 $\chi(\mathcal{A}(\Phi^+, \Psi, m), t) = \chi(\mathcal{A}(\Psi), t - mh).$

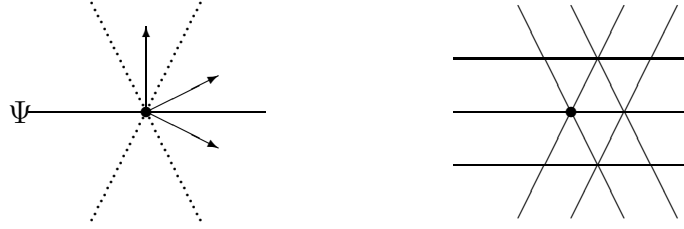


Figure 2: Subarrangement Ψ and $\mathcal{A}(A_2, \Psi, 1)$

Note that this family contains both extended Shi and Catalan arrangements. In fact, taking $\Psi = \Phi^+$ we have the extended Catalan arrangement $\mathcal{A}(\Phi^+, \Phi^+, m) = \mathcal{A}(\Phi^+)^{[-m, m]}$ and taking $\Psi = \phi(\text{empty arrangement})$, we have the extended Shi arrangement $\mathcal{A}(\Phi, \phi, m) = \mathcal{A}(\Phi^+)^{[1-m, m]}$.

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